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Stability and Bifurcation of Traveling Wave Solutions of Nerve Axon Type Equations

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1. STATEMENT AND DISCUSSION OF RESULTS

In this paper we will study the stability and bifurcation of traveling wave solutions of the problem

$$Y_t = EY_{\xi\xi} + F(Y, \beta), \qquad -\infty < \xi < \infty,\ t \geqslant 0 \tag{1.1a}$$

$$\lim_{|\xi| \to \infty} Y(\xi, t) = 0, \qquad t \geqslant 0, \tag{1.1b}$$

where β is a parameter in the real Banach space $\mathscr{B}$, $F\colon \mathbf{R}^n \times \mathscr{B} \to \mathbf{R}^n$ is a twice continuously differentiable function, $F(0, \beta) \equiv 0$, and $E = \left(\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where D is a real, constant, $p \times p$, nonsingular matrix. For D equal to the one by one identity matrix, (1.1a) is the nerve axon equation studied by Evans [4–7]. This equation is, in turn, a generalization of the FitzHugh–Nagumo equation,

$$\begin{aligned} v_t &= v_{\xi\xi} + f(v) - w \\ w_t &= bv, \end{aligned} \tag{1.2}$$

and the Hodgkin–Huxley equation. In (1.2)

$$f(v) = v(v-a)(1-v).$$

To put (1.2) in the form of (1.1a) we would let $\beta = (a, b)$.

For $u \in \mathbf{R}^n$ and $\beta \in \mathscr{B}$ we define u_1, $F_1(u_1, u_2, \beta) \in \mathbf{R}^p$ and u_2, $F_2(u_1, u_2, \beta) \in \mathbf{R}^{n-p}$ by

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \text{and} \qquad F(u, \beta) = \begin{pmatrix} F_1(u_1, u_2, \beta) \\ F_2(u_1, u_2, \beta) \end{pmatrix}.$$

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If, for some $\beta \in \mathscr{B}$, $Y(\xi, t)$ is a solution of (1.1a), (1.1b) and $Y(\xi, t) = u(\xi + ct)$ for some nonzero real number c and some continuously differentiable function $u(x)$ such that $u(x) \not\equiv 0$ and $u_1''(x)$ exists for all $x \in \mathbf{R}$, then we call $Y(\xi, t)$ a *traveling wave solution* of (1.1a), (1.1b) of speed c. In this case we must have

$$G(u, \beta, c) = 0 \tag{1.3a}$$

$$\lim_{|x| \to \infty} u(x) = 0, \tag{1.3b}$$

where $G(u, \beta, c)$ stands for

$$Eu'' - cu' + F(u, \beta). \tag{1.4}$$

It is not hard to prove that any solution, $u(x)$, of (1.3a), (1.3b) with $c \neq 0$ and $\beta \in \mathscr{B}$ must satisfy

$$\lim_{|x| \to \infty} u'(x) = 0 \qquad \text{and} \qquad \lim_{|x| \to \infty} u_1''(x) = 0. \tag{1.3c}$$

We let $G_u(u, \beta, c)$ stand for the expression obtained by formally differentiating (1.4) with respect to u. Thus $G_u(u, \beta, c)$ stands for

$$E\frac{d^2}{dx^2} - c\frac{d}{dx} + F_u(u, \beta).$$

Suppose, for some $\beta_0 \in \mathscr{B}$ and $c_0 \in \mathbf{R} - \{0\}$,

$$Y_0(\xi, t) = u_0(\xi + c_0 t) \tag{1.5}$$

is a traveling wave solution of (1.1a), (1.1b). Then substituting $(u, \beta, c) = (u_0, \beta_0, c_0)$ in (1.3a), differentiating the resulting equation with respect to x, and noting (1.3c), we see that $\phi = u_0'$ is a nontrivial solution of

$$G_u(u_0, \beta_0, c_0)\phi = \lambda\phi \tag{1.6a}$$

$$\lim_{|x| \to \infty} \phi(x) = 0 \tag{1.6b}$$

with $\lambda = 0$.

We define (1.5) to be a *linearly stable* solution of (1.1a), (1.1b) with $\beta = \beta_0$ if $\lambda = 0$ is a *simple eigenvalue* of (1.6a), (1.6b) (that is, the space of solutions, ϕ, of (1.6a), (1.6b) with $\lambda = 0$ is spanned by u_0' and the problem

$$G_u(u_0, \beta_0, c_0)\phi = u_0' \tag{1.7a}$$

$$\lim_{|x| \to \infty} \phi(x) = 0 \tag{1.7b}$$

has no solutions), and the supremum of the real parts of the nonzero eigenvalues of (1.6a), (1.6b) is negative. We say it is *linearly unstable* if (1.6a), (1.6b) have an eigenvalue with positive real part.

If $\hat{u}(x)$ is a translate of $u_0(x)$, that is, $\hat{u}(x) = u_0(x + x_0)$ for some $x_0 \in \mathbf{R}$, then clearly $\hat{Y}(\xi, t) = \hat{u}(\xi + c_0 t)$ is a traveling wave solution of (1.1a), (1.1b) with $\beta = \beta_0$ and $\hat{Y}(\xi, t)$ is linearly stable (unstable) if and only if $Y_0(\xi, t)$ is. In the case D is the one by one identity matrix, Evans [6] has proved that if (1.5) is a linearly stable solution of (1.1a), (1.1b) with $\beta = \beta_0$ then there exists $\delta > 0$ such that if $Y(\xi, t)$ is a solution of (1.1a), (1.1b) with $\beta = \beta_0$ such that for some $x_0 \in \mathbf{R}$

$$\| Y(\xi, 0) - u_0(\xi + x_0)\|_\infty < \delta$$

then there exists $x_1 \in \mathbf{R}$ such that

$$\| Y(\xi, t) - u_0(\xi + c_0 t + x_1)\|_\infty \to 0$$

as $t \to \infty$.

Under the change of variables $y_1 = u_1$, $y_2 = u_2$, $y_3 = u_1'$, and

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

(1.3a), (1.3b) are equivalent to

$$y' = \mathscr{F}(y, \beta, c) \tag{1.8a}$$

$$\lim_{|x| \to \infty} y(x) = 0, \tag{1.8b}$$

where $\mathscr{F}: \mathbf{R}^{n+p} \times \mathscr{B} \times \mathbf{R} \to \mathbf{R}^{n+p}$ is given by

$$\mathscr{F}(y, \beta, c) = \begin{pmatrix} y_3 \\ (1/c) F_2(y_1, y_2, \beta) \\ cD^{-1} y_3 - D^{-1} F_1(y_1, y_2, \beta) \end{pmatrix}.$$

Corresponding to the nontrivial solution (u_0, β_0, c_0) of (1.3a), (1.3b) there is a nontrivial solution (y_0, β_0, c_0) of (1.8a), (1.8b). Hence $\mathscr{F}_y(0, \beta_0, c_0)$ has at least one eigenvalue with nonnegative real part and at least one eigenvalue with nonpositive real part.

We make the following hypothesis

(H_1). *There is a nonempty open subset Ω of $\mathscr{B} \times (\mathbf{R} - \{0\})$ such that, for all $(\beta, c) \in \Omega$, $\mathscr{F}_y(0, \beta, c)$ has one simple positive eigenvalue, $\mu(\beta, c)$, and all the other eigenvalues of $\mathscr{F}_y(0, \beta, c)$ have negative real part.*

For the FitzHugh–Nagumo equation $\Omega = \{(a, b, c)\colon a, b$ and c are positive$\}$ satisfies Hypothesis (H_1). Likewise, for the Hodgkin–Huxley equation there exists Ω which satisfies Hypothesis (H_1) and contains the values of (β, c) of physical importance.

For $(\beta, c) \in \Omega$, we can choose an eigenvector $v(\beta, c)$ of length one which corresponds to the positive eigenvalue of $\mathscr{F}_y(0, \beta, c)$ and varies continuously with (β, c). Let S be the set of all (β, c) in Ω such that (1.1a), (1.1b) have a traveling wave solution of speed c. Then $S = S^+ \cup S^-$, where S^+ (resp. S^-) is the set of all (β, c) in Ω such that (1.1a), (1.1b) has a traveling wave solution, $u(\xi + ct)$, and

$$\begin{pmatrix} u(x) \\ u_1'(x) \end{pmatrix} e^{-\mu(\beta, c)x}$$

tends to $v(\beta, c)$ (resp. $-v(\beta, c)$) as $x \to -\infty$. By Hypothesis (H_1), for each (β, c) in S^+ (resp. S^-) there exists exactly one traveling wave solution $u_+(\xi + ct, \beta, c)$ (resp. $u_-(\xi + ct, \beta, c)$) of (1.1a), (1.1b) satisfying the condition for (β, c) to be in S^+ (resp. S^-). Furthermore, if $u(\xi + ct)$ is a traveling wave solution of (1.1a), (1.1b) corresponding to some $(\beta, c) \in S$ then $u(x)$ must be a translate of either $u_+(x, \beta, c)$ or $u_-(x, \beta, c)$. In what follows we restrict our attention to S^+. However, the same result will hold for S^-.

By Hypothesis (H_1) we have

LEMMA 1.1. *For $(\beta, c) \in S^+$, the space of solutions, ϕ, of*

$$G_u(u_+(x, \beta, c), \beta, c)\phi = 0$$
$$\lim_{|x| \to \infty} \phi(x) = 0$$

is spanned by $u'_+(x, \beta, c)$.

It is not hard to prove

THEOREM 1.1. *If (β_0, c_0) is a fixed point of S^+, $u_0(x) = u_+(x, \beta_0, c_0)$, and $\lambda = 0$ is a simple eigenvalue of* (1.6a), (1.6b), *then there exists $r > 0$, a $\mathscr{B} \times \mathbf{R}$-neighborhood, $\mathscr{B}_0 \times \mathbf{R}_0$, of (β_0, c_0), and a continuously differentiable function $c\colon \mathscr{B}_0 \to \mathbf{R}_0$ with $c(\beta_0) = c_0$ such that S^+ intersect $\mathscr{B}_0 \times \mathbf{R}_0$ is $\{(\beta, c(\beta))\colon \beta \in \mathscr{B}_0\}$ and, for $\beta \in \mathscr{B}_0$,*

(i) *zero is a simple eigenvalue of*

$$G_u(u_+(x, \beta, c(\beta)), \beta, c(\beta))\phi = \lambda\phi \tag{1.9a}$$

$$\lim_{|x| \to \infty} \phi(x) = 0 \tag{1.9b}$$

and

(ii) (1.9a), (1.9b) *have no nontrivial solution*, ϕ, *for* $\lambda \in \mathbb{C}$ *and* $0 < |\lambda| < r$.

We say a point (β, c) of S^+ is *stable* (*unstable*) if the corresponding traveling wave solution, $u_+(\xi + ct, \beta, c)$ of (1.1a), (1.1b), is linearly stable (linearly unstable). By Lemma 1.1 and Theorem 1.1, if (β_0, c_0) is a point of S^+ where a bifurcation of S^+ takes place or a change of stability of points in S^+ takes place due to an eigenvalue crossing the imaginary axis at zero, then $\lambda = 0$ must be an eigenvalue of (1.6a), (1.6b) of geometric multiplicity one and algebraic multiplicity greater than or equal to two. In particular, $\lambda = 0$ *cannot be a semisimple eigenvalue of* (1.6a), (1.6b).

Our main result, which deals with the generic case that $\lambda = 0$ is an eigenvalue of algebraic multiplicity two, is

THEOREM 1.2. *If* (β_0, c_0) *is a fixed point of* S^+, $u_0(x) = u_+(x, \beta_0, c_0)$, *and* $\lambda = 0$ *is an eigenvalue of* (1.6a), (1.6b) *of algebraic multiplicity* 2 (*that is, there exists a solution* $\phi = \hat{u}$ *of* (1.7a), (1.7b) *however the problem* (1.7a), (1.7b) *with* u_0' *replaced with* $\hat{u}$ *has no solution*) *then there exists* $r > 0$, *a* $\mathscr{B} \times \mathbf{R}$-*neighborhood*, $\mathscr{B}_0 \times \mathbf{R}_0$, *of* (β_0, c_0) *and continuously differentiable functions* $\lambda: \mathscr{B}_0 \times \mathbf{R}_0 \to \{\lambda \in \mathbf{R} : |\lambda| < r\}$ *and* $\psi: \mathscr{B}_0 \times \mathbf{R}_0 \to \mathbf{R}$ *with* $\lambda(\beta_0, c_0) = 0$, $\psi(\beta_0, c_0) = 0$, *and* $\psi_c(\beta_0, c_0) = 0$ *such that* S^+ *intersect* $\mathscr{B}_0 \times \mathbf{R}_0$ *is*

$$\{(\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0 : \psi(\beta, c) = 0\}$$

and for $(\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0$ *with* $\psi(\beta, c) = 0$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| < r$ *we have*

(i) *the problem*

$$G_u(u_+(x, \beta, c), \beta, c)\phi = \lambda\phi \tag{1.10a}$$

$$\lim_{|x| \to \infty} \phi(x) = 0 \tag{1.10b}$$

has a nontrivial solution, ϕ, *if and only if* λ *is zero or* $\lambda(\beta, c)$;

(ii) $\operatorname{sgn} \lambda(\beta, c) = \operatorname{sgn} \psi_c(\beta, c)$; *and*

(iii) $\lambda(\beta, c) \neq 0$ *if and only if* 0 *and* $\lambda(\beta, c)$ *are both simple eigenvalues of* (1.10a), (1.10b).

Furthermore, if for some $\beta \in \mathscr{B}$, *the problem obtained from* (1.7a), (1.7b) *by replacing* u_0' *with* $F_\beta(u_0(x), \beta_0)\beta$ *has no solution*, ϕ, *then, in some* $\mathscr{B} \times \mathbf{R}$-*neighborhood of* (β_0, c_0), *the zero set of* ψ *is given by a continuously differentiable, codimension one submanifold*, M, *of* $\mathscr{B} \times \mathbf{R}$ *such that for all* $(\beta, c) \in M$, *we have that the tangent space to* M *at* (β, c) *is parallel to the c-axis if and only if* $\psi_c(\beta, c) = 0$.

Part (ii) of Theorem 1.2 is known as the *principle of exchange of stability* (PES). Since the derivatives at adjacent zeros of a real valued function of a real variable cannot both be positive and cannot both be negative, the PES implies, for each fixed $\beta_1 \in \mathscr{B}_0$, that adjacent points of S^+ intersect $\{\beta_1\} \times \mathbf{R}_0$ cannot both be stable and cannot both be unstable (provided, of course, each nonzero eigenvalue of (1.6a), (1.6b) has negative real part). It is known that the PES does not always hold for bifurcation from an eigenvalue of algebraic multiplicity two and geometric multiplicity one. However, it holds in our case because we have the extra parameter c. The PES also holds for Hopf bifurcation because one can phrase the problem so that ω, the period of the bifurcating solution, is an extra parameter. However, in the case of Hopf bifurcation, the bifurcation is from an eigenvalue of algebraic and geometric multiplicity two. So the eigenvalue is semisimple. Also, for Hopf bifurcation, the PES does *not* take place in the (β, ω)-plane. Theorem 1.2 states, for our problem, that *the PES holds for bifurcation from an eigenvalue which is not semisimple and that the PES holds in the* (β, c) *plane.*

We will prove Theorem 1.2 by using a modification of Weinberger's proof in [14, Sect. 4]. Since $\lambda = 0$ is not a semisimple eigenvalue of (1.6a), (1.6b) it is not necessary to introduce the auxiliary parameter denoted by α in [14] and one can obtain the PES in the (β, c)-plane rather than in the (β, α)-plane. The latter is important because in applications the family of traveling wave solutions of (1.1a), (1.1b) is usually graphed by graphing S^+ in the (β, c) plane.

For example, in the case (1.1a) is the FitzHugh equation (1.2), for each $b > 0$, let S_b^+ be the set of all (a, c) such that $(a, b, c) \in S^+$. According to numerical results [8], the graph of the set, S_{b1}^+, of all points in S_b^+ corresponding to single pulse traveling wave solutions of (1.1a), (1.1b) is given in Fig. 1, and if (a, c) is a point on S_{b1}^+ with $c > \hat{c}$ $(c < \hat{c})$ then (a, c) is stable (unstable). (Actually these numerical results are not true for both

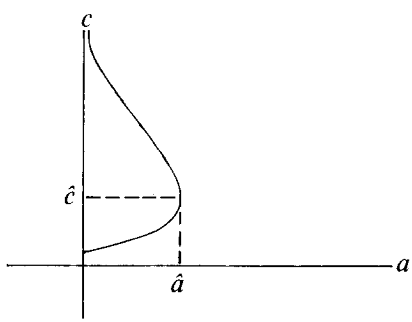

FIG. 1. Graph of S_{b1}^+.

S^+ and S^- but only for S^+ or S^-. By replacing $v(\beta, c)$ with its negative, if necessary, we can assume they are true for S^+.)

Some of these numerical results, in the case b is sufficiently small, have been proved analytically [1, 2, 9, 10, 12]. Also, Rinzel and Keller [13] have analytically verified most of these numerical results in the case the cubic, $f(v)$, in (1.2) is replaced with a certain piecewise linear function.

Let (H_2) *be the hypothesis that*

(i) *the graph of* S_{b1}^+ *is given by Fig.* 1; *and*

(ii) *only generic changes of stability take place in* S_{b1}^+.

That is, if (a_0, c_0) is a point in S_{b1}^+ where a change of stability takes place, then either zero is an eigenvalue of (1.6a), (1.6b) of algebraic multiplicity two and (1.6a), (1.6b) have no nonzero eigenvalues on the imaginary axis, or, for some $\omega_0 > 0$, the only eigenvalues of (1.6a), (1.6b) on the imaginary axis are 0, $i\omega_0$, and $-i\omega_0$, all of which are simple.

By Theorem 1.2 we have

COROLLARY. *If Hypothesis* (H_2) *holds, there exists a stable point in* S_{b1}^+ *with* $c > c_0$, *and only generic changes of stability of the first type described in the last paragraph take place in* S_{b1}^+, *then all points on* S_{b1}^+ *with* $c > \hat{c}$ *are not unstable, and all points on* S_{b1}^+ *with* $c < \hat{c}$ *are not stable.*

In a later paper we plan to fill in the gap in this corollary by proving a Hopf bifurcation analog of Theorem 1.2 when (β_0, c_0) is a point in S^+ where a generic change of stability takes place of the type excluded in the corollary. We have not done this here because it turns out that the operator for the Hopf case, corresponding to the above $G_u(u_0, \beta_0, c_0)$, is not Fredholm and thus methods quite different from those used here will be required.

In Section 2 we prove general abstract versions of Theorems 1.1 and 1.2. In Section 3 we prove Theorems 1.1 and 1.2 by showing the problem (1.1a), (1.1b) satisfies the assumptions made in Section 2.

2. ABSTRACT VERSIONS OF THEOREMS 1.1 AND 1.2

The main results of this section are Theorems 2.1 and 2.2, which are abstract versions of Theorems 1.1 and 1.2, respectively. This section can be read independently of Section 1.

We define some notation and terminology. If L is a linear operator then $N(L)$ and $R(L)$ stand for the null space and range of L, respectively. Suppose L and K are bounded linear operators from a Banach space U into a Banach space V and λ is a complex number. We say λ is a *K-simple* *eigenvalue* of L if $N(L-\lambda K)$ is one dimensional, $R(L-\lambda K)$ has codimension one in V, and

$$R(L-\lambda K) \cap K(N(L-\lambda K)-\{0\}) = \emptyset.$$

Let $(\mathrm{H_s})$ be the hypothesis that

(i) A and B are real Banach spaces;

(ii) U and V are real Banach spaces of functions $u: \mathbf{R} \to A$ and $v: \mathbf{R} \to B$, respectively, where addition and scalar multiplication are done pointwise, and, for each $u \in U$, $\lim_{x \to -\infty} u(x)$ and $\lim_{x \to \infty} u(x)$ both exist;

(iii) for each $\xi \in \mathbf{R}$ the translation operators $S_\xi: U \to U$ and $T_\xi: V \to V$ given by $(S_\xi u)(x) = u(x+\xi)$ and $(T_\xi v)(x) = v(x+\xi)$ are bounded;

(iv) for each $u \in U$ and $v \in V$ there exist positive constants M_u and M_v such that $\|S_\xi u\| < M_u$ and $\|T_\xi v\| < M_v$ for all $\xi \in \mathbf{R}$;

(v) if for some $a \in A$ and some sequence $\{u_n\}_{n=0}^{\infty}$ in U we have $\lim_{n\to\infty} u_n = u_0$ and, for each real number x, $\lim_{n\to\infty} u_n(x) = a$, then $u_0(x) = a$ for all x.

Note that (iv) and the principle of uniform boundedness imply the existence of a positive constant M such that

$$\|S_\xi\| < M \qquad \text{and} \qquad \|T_\xi\| < M \tag{2.1}$$

for all $\xi \in \mathbf{R}$.

If Hypothesis $(\mathrm{H_s})$ holds, $u \in U$, and

$$\lim_{\xi \to 0} \frac{S_\xi u - u}{\xi} \tag{2.2}$$

exists (where the limit is taken in the topology of U), then we denote (2.2) by $\mathscr{D}_x u$.

Let (H) be the hypothesis that Hypothesis $(\mathrm{H_s})$ holds and

(i) $K, \kappa\delta: U \to V$ are bounded linear operators such that for all $\xi \in \mathbf{R}$ we have $KS_\xi = T_\xi K$ and $\kappa\delta S_\xi = T_\xi \kappa\delta$, and for all $u \in U$ such that (2.2) exists we have $\kappa\delta_u = K\mathscr{D}_x u$;

(ii) $\mathscr{B}$ is a real Banach space, $g: U \times \mathscr{B} \to V$ is a continuously differentiable function, and $g(S_\xi u, \beta) = T_\xi g(u, \beta)$ for all $\xi \in \mathbf{R}$ and $(u, \beta) \in U \times \mathscr{B}$;

(iii) $G: U \times \mathscr{B} \times \mathbf{R} \to V$ is defined by

$$G(u, \beta, c) = g(u, \beta) - c\kappa\, \delta u$$

and for all (u, β, c) in the zero set of G the limit (2.2) exists.

In part (i) of Hypothesis (H), $\kappa\delta$ does not stand for the composition of κ with δ, but rather it stands for a single operator which is denoted by the juxtaposition of the two symbols κ and δ. Each of these symbols standing alone has no meaning.

The following two statements follow directly from Hypothesis (H). For all $(u, \beta, c) \in U \times \mathscr{B} \times \mathbf{R}$ and $\xi \in \mathbf{R}$,

$$G(S_\xi u, \beta, c) = T_\xi G(u, \beta, c). \tag{2.3}$$

If (u_0, β_0, c_0) is in the zero set of G then $\xi \to S_\xi u_0$ is a differentiable curve in U and differentiating (2.3) with respect to ξ at $\xi = 0$ and $(u, \beta, c) = (u_0, \beta_0, c_0)$ we get

$$G_u(u_0, \beta_0, c_0)(\mathscr{D}_x u_0) = 0. \tag{2.4}$$

If Hypothesis (H) holds, (u_1, β_1, c_1) and (u_2, β_2, c_2) are two points in $U \times \mathscr{B} \times \mathbf{R}$, and $(u_2, \beta_2, c_2) = (S_\xi u_1, \beta_1, c_1)$ for some $\xi \in \mathbf{R}$, then we say (u_2, β_2, c_2) is a translate of (u_1, β_1, c_1).

LEMMA 2.1. *Suppose Hypothesis* (H) *holds,* (u_0, β_0, c_0) *is in the zero set of* G, *and* $v^*\colon V \to \mathbf{R}$ *is a bounded linear functional with* $v^* K \mathscr{D}_x u_0 > 0$. *Let* $\hat{U} = \{u \in U\colon v^* K u = v^* K u_0\}$. *Then* (i) *for each* $\hat{U} \times \mathscr{B} \times \mathbf{R}$*-neighborhood,* $\hat{\mathcal{O}}$, *of* (u_0, β_0, c_0), *there is a* $U \times \mathscr{B} \times \mathbf{R}$*-neighborhood,* $\mathcal{O}$, *of* (u_0, β_0, c_0) *such that all solutions, in* $\mathcal{O}$, *of* $G = 0$ *are translates of solutions, in* $\hat{\mathcal{O}}$, *of* $G = 0$; *and* (ii) *there exists a* $\hat{U} \times \mathscr{B} \times \mathbf{R}$*-neighborhood of* (u_0, β_0, c_0) *which does not contain two different solutions of* $G = 0$ *such that one is a translate of the other.*

Proof. Suppose there exists a $\hat{U} \times \mathscr{B} \times \mathbf{R}$-neighborhood, $\hat{\mathcal{O}}$, of (u_0, β_0, c_0) such that part (i) of the lemma is false. Then there exists $\varepsilon > 0$ and a sequence $\{(u_n, \beta_n, c_n)\}$ in the zero set of G such that $\lim_{n \to \infty}(u_n, \beta_n, c_n) = (u_0, \beta_0, c_0)$ and for each n the differentiable curve $\xi \to S_\xi u_n$ does not intersect

$$\{u \in \hat{U}\colon \|u - u_0\| < \varepsilon\}.$$

Since $(d/d\xi)(v^* K S_\xi u_0)|_{\xi=0} = v^* K \mathscr{D}_x u_0 > 0$, we have for some $\xi_0 > 0$ that $\xi(v^* K S_\xi u_0 - v^* K u_0) > 0$ for $0 < |\xi| < \xi_0$. Hence for each positive integer $k > 1/\xi_0$ there exists n_k such that $\|u_{n_k} - u_0\| < 1/k$ and

$$v^* K S_{-1/k} u_{n_k} < v^* K u_0 < v^* K S_{1/k} u_{n_k}.$$

Thus, for $k > 1/\xi_0$, there exists $\xi_k \in (-1/k, 1/k)$ such that $v^* K S_{\xi_k} u_{n_k} = v^* K u_0$ and therefore $S_{\xi_k} u_{n_k} \in \hat{U}$. Since, by (2.1),

$$S_{\xi_k} u_{n_k} - u_0 = S_{\xi_k}(u_{n_k} - u_0) + (S_{\xi_k} u_0 - u_0) \to 0$$

as $k \to \infty$, we have a contradiction and part (i) of the lemma is established.

Suppose part (ii) of the lemma is false. Then there exist sequences $\{(u_n, \beta_n, c_n)\}_{n=1}^{\infty}$ and $\{(w_n, \beta_n, c_n)\}_{n=1}^{\infty}$ in $\hat{U} \times \mathscr{B} \times \mathbf{R}$ intersect the zero set of G, which both converge to (u_0, β_0, c_0), and a sequence of positive real numbers $\{\xi_n\}$ such that

$$w_n = S_{\xi_n} u_n \tag{2.5}$$

for all n. By taking a subsequence of necessary we can assume for some $\xi_0 \in [0, \infty]$ that $\lim_{n \to \infty} \xi_n = \xi_0$.

Case I. Suppose $\xi_0 = 0$. Since, for each fixed n, the differentiable curve $\xi \to v^* K S_\xi u_n$ has the value $v^* K u_0$ at $\xi = 0$ and $\xi = \xi_n$, there exists, by the mean value theorem, $\zeta_n \in (0, \xi_n)$ such that

$$0 = v^* K S_{\zeta_n} \mathscr{D}_x u_n = v^* T_{\zeta_n} \kappa \delta u_n. \tag{2.6}$$

Since, by (2.1),

$$T_{\zeta_n} \kappa \delta u_n - \kappa \delta u_0 = T_{\zeta_n}(\kappa \delta u_n - \kappa \delta u_0) + \kappa \delta (S_{\zeta_n} u_0 - u_0) \to 0$$

as $n \to \infty$, we have

$$\lim_{n \to \infty} v^* T_{\zeta_n} \kappa \delta u_n = v^* \kappa \delta u_0 = v^* K \mathscr{D}_x u_0 > 0$$

which contradicts (2.6).

Case II. Suppose $\xi_0 \in (0, \infty)$. Since, by (2.1),

$$S_{\xi_n} u_n - S_{\xi_0} u_0 = S_{\xi_n}(u_n - u_0) + (S_{\xi_n} u_0 - S_{\xi_0} u_0) \to 0$$

as $n \to \infty$, we have, letting $n \to \infty$ in (2.5) that $u_0 = S_{\xi_0} u_0$. Hence, for all positive integers n and all $x \in \mathbf{R}$, we have $u_0(x) = u_0(x + n\xi_0)$. Letting $n \to \infty$, we get for all $x \in \mathbf{R}$ that $u_0(x) = \lim_{\xi \to \infty} u_0(\xi)$. Hence $\mathscr{D}_x u_0 = 0$—a contradiction.

Case III. Suppose $\xi_0 = \infty$. Then, by (2.5) and (2.1),

$$S_{\xi_n} u_0 - u_0 = S_{\xi_n}(u_0 - u_n) + (w_n - u_0) \to 0$$

as $n \to \infty$. Thus, by Hypothesis ($\mathrm{H_s}$), part (v), $u_0(x) = \lim_{\xi \to \infty} u_0(\xi)$ for all $x \in \mathbf{R}$. Hence $\mathscr{D}_x u_0 = 0$—a contradiction.

Let Hypothesis (H_3) be the hypothesis that Hypothesis (H) holds, (u_0, β_0, c_0) is a fixed solution of $G = 0$ with $\mathscr{D}_x u_0 \neq 0$, and zero is a K-simple eigenvalue of $G_u(u_0, \beta_0, c_0)$.

Let Hypothesis (H_4) be Hypothesis (H_3) with the condition that zero is a K-simple eigenvalue of $G_u(u_0, \beta_0, c_0)$ replaced with the conditions that

$N(G_u(u_0, \beta_0, c_0)) = \operatorname{span}\{\mathscr{D}_x u_0\}$, $R(G_u(u_0, \beta_0, c_0))$ has codimension one in V, and there exists $u_2 \in U$ such that $K(\mathscr{D}_x u_0) = G_u(u_0, \beta_0, c_0)\, u_2$ and $Ku_2 \notin R(G_u(u_0, \beta_0, c_0))$.

If either Hypothesis (H_3) or (H_4) holds let $u_1 = \mathscr{D}_x u_0$ and $T_0 = G_u(u_0, \beta_0, c_0)$.

If Hypothesis (H_3) holds, let v_1^* be the unique bounded linear functional on V such that $v_1^* T_0 = 0$ and $v_1^* Ku_1 = 1$.

If Hypothesis (H_4) holds, let v_1^* and v_2^* be the unique bounded linear functionals on V such that part (ii) of Lemma 01. in the Appendix holds.

When either Hypothesis (H_3) or (H_4) holds we define

$$\hat{U} = \{u \in U \colon v_1^* Ku = v_1^* Ku_0\}.$$

By Lemma 2.1, in order to find all solutions of $G = 0$ in a $U \times \mathscr{B} \times \mathbf{R}$ neighborhood of (u_0, β_0, c_0) it suffices to find all solutions of $G = 0$ in a $\hat{U} \times \mathscr{B} \times \mathbf{R}$-neighborhood of (u_0, β_0, c_0).

THEOREM 2.1. *If Hypothesis* (H_3) *holds then there exists* $r > 0$, *a* $\hat{U} \times \mathscr{B} \times \mathbf{R}$*-neighborhood,* $\hat{U}_0 \times \mathscr{B}_0 \times \mathbf{R}_0$, *of* (u_0, β_0, c_0), *and continuously differentiable functions* $u \colon \mathscr{B}_0 \to \hat{U}_0$ *and* $c \colon \mathscr{B}_0 \to \mathbf{R}_0$ *with* $u(\beta_0) = u_0$ *and* $c(\beta_0) = c_0$ *such that*

$$\{(u, \beta, c) \in \hat{U}_0 \times \mathscr{B}_0 \times \mathbf{R}_0 \colon G(u, \beta, c) = 0\} = \{(u(\beta), \beta, c(\beta)) \colon \beta \in \mathscr{B}_0\} \tag{2.7}$$

and, for $\beta \in \mathscr{B}_0$, *zero is a K-simple eigenvalue of* $G_u(u(\beta), \beta, c(\beta))$ *and*

$$(G_u(u(\beta), \beta, c(\beta)) - \lambda K) \colon U \to V$$

is an isomorphism for $\lambda \in \mathbb{C}$ *and* $0 < |\lambda| < r$.

Proof. The partial derivative of the left side of the system

$$G(u, \beta, c) = 0$$

$$v_1^* Ku - v_1^* Ku_0 = 0$$

at $(u, \beta, c) = (u_0, \beta_0, c_0)$ with respect to (u, c) is

$$\begin{pmatrix} T_0 & -Ku_1 \\ v_1^* K & 0 \end{pmatrix}. \tag{2.8}$$

Since (2.8) is an isomorphism from $U \times \mathbf{R}$ onto $V \times \mathbf{R}$ we have by the implicit function theorem that the part of Theorem 2.1 up to and including (2.7) is true.

Since $v_1^* K\mathscr{D}_x u_0 = 1$, $\kappa\delta$ is bounded, and, for $\beta \in \mathscr{B}_0$, $v_1^* \kappa\delta u(\beta) = v_1^* K\mathscr{D}_x u(\beta)$ we have, by taking $\mathscr{B}_0$ smaller if necessary, that $v_1^* KD_x u(\beta) \neq 0$

for $\beta \in \mathscr{B}_0$. Hence $\mathscr{D}_x u(\beta) \neq 0$ for $\beta \in \mathscr{B}_0$. Thus, since $\mathscr{D}_x u(\beta)$ is in the null space of $G_u(u(\beta), \beta, c(\beta))$, by taking $\mathscr{B}_0$ smaller if necessary, the part of Theorem 2.1 after (2.7) follows from Crandall and Rabinowitz [3, Lemma 1.3].

THEOREM 2.2. *If Hypothesis* (H_4) *holds then there exists* $r > 0$, *a* $\hat{U} \times \mathscr{B} \times \mathbf{R}$*-neighborhood,* $\hat{U}_0 \times \mathscr{B}_0 \times \mathbf{R}_0$, *of* (u_0, β_0, c_0), *and continuous functions* $u: \mathscr{B}_0 \times \mathbf{R}_0 \to \hat{U}_0$, $\phi: \mathscr{B}_0 \times \mathbf{R}_0 \to \mathbf{R}$, *and* $\lambda: \mathscr{B}_0 \times \mathbf{R}_0 \to \{\lambda \in \mathbf{R}: |\lambda| < r\}$ *with* u *and* ϕ *continuously differentiable,* $u(\beta_0, c_0) = u_0$, $\phi(\beta_0, c_0) = 0$, $\phi_c(\beta_0, c_0) = 0$, *and* $\lambda(\beta_0, c_0) = 0$ *such that the zero set of* G *intersect* $\hat{U}_0 \times \mathscr{B}_0 \times \mathbf{R}_0$ *is*

$$\{(u(\beta, c), \beta, c): (\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0 \text{ and } \phi(\beta, c) = 0\} \tag{2.9}$$

and for $(\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0$ *with* $\phi(\beta, c) = 0$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| < r$ *we have*

(i) $(G_u(u(\beta, c), \beta, c) - \lambda K): U \to V$ *is an isomorphism if and only if* $\lambda \notin \{0, \lambda(\beta, c)\}$;

(ii) $\operatorname{sgn} \lambda(\beta, c) = \operatorname{sgn} \phi_c(\beta, c)$; *and*

(iii) $\lambda(\beta, c) \neq 0$ *if and only if* 0 *and* $\lambda(\beta, c)$ *are both* K*-simple eigenvalues of* $G_u(u(\beta, c), \beta, c)$.

Furthermore, if $R(G_\beta(u_0, \beta_0, c_0)) \not\subset R(T_0)$, *then, in some* $\mathscr{B} \times \mathbf{R}$*-neighborhood of* (β_0, c_0), *the zero of set of* ϕ *is given by a continuously differentiable, codimension one submanifold,* M, *of* $\mathscr{B} \times \mathbf{R}$ *such that for all* $(\beta, c) \in M$ *we have that the tangent space to* M *at* (β, c) *is parallel to the c-axis if and only if* $\phi_c(\beta, c) = 0$.

Proof. Define $f: \mathscr{B} \times \mathbf{R} \times U \times \mathbf{R} \to V \times R$ by

$$f(\beta, c, u, \phi) = \begin{pmatrix} G(u, \beta, c) - \phi K u_2 \\ v_1^* K u - v_1^* K u_0 \end{pmatrix}.$$

Since

$$V = R(T_0) \oplus \operatorname{span}\{K u_2\} \tag{2.10}$$

we have

$$f_{(u,\phi)}(\beta_0, c_0, u_0, 0) = \begin{pmatrix} T_0 & -K u_2 \\ v_1^* K & 0 \end{pmatrix} \tag{2.11}$$

is an isomorphism. Thus, by the implicit function theorem, there exists a neighborhood, $\mathscr{B}_0 \times \mathbf{R}_0 \times U_0 \times \mathbf{R}_\phi$, of $(\beta_0, c_0, u_0, 0)$ and continuously differentiable functions $u: \mathscr{B}_0 \times \mathbf{R}_0 \to U_0$ and $\phi: \mathscr{B}_0 \times \mathbf{R}_0 \to \mathbf{R}_\phi$, with

$u(\beta_0, c_0) = u_0$ and $\phi(\beta_0, c_0) = 0$ such that the zero set of f intersect $\mathscr{B}_0 \times \mathbf{R}_0 \times U_0 \times \mathbf{R}_\phi$ is

$$\{(\beta, c, u(\beta, c), \phi(\beta, c)) \colon (\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0\}.$$

Let $\hat{U}_0 = U_0 \cap \hat{U}$. Clearly the range of the function u is contained in $\hat{U}_0$ and the zero set of G intersect $\hat{U}_0 \times \mathscr{B}_0 \times \mathbf{R}_0$ is (2.9). Differentiating

$$f(\beta, c, u(\beta, c), \phi(\beta, c)) \equiv 0 \tag{2.12}$$

with respect to c we get

$$f_{(u,\phi)}(\beta, c, u(\beta, c), \phi(\beta, c)) \begin{pmatrix} u_c(\beta, c) \\ \phi_c(\beta, c) \end{pmatrix} = \begin{pmatrix} \kappa\delta u(\beta, c) \\ 0 \end{pmatrix}. \tag{2.13}$$

Since (2.11) is an isomorphism and maps $(u_2, 0)^T$ to $(Ku_1, 0)^T$ we have by (2.13) with $(\beta, c) = (\beta_0, c_0)$ that $u_c(\beta_0, c_0) = u_2$ and $\phi_c(\beta_0, c_0) = 0$.

Define $\mathscr{F} \colon \mathscr{B}_0 \times \mathbf{R}_0 \times V^* \times \mathbf{R}^2 \to U^* \times \mathbf{R}^2$ by

$$\mathscr{F}(\beta, c, v^*, \lambda, \sigma) = \begin{pmatrix} v^*[T(\beta, c) - \lambda K] - \sigma v_1^* K \\ v^* K u_2 - 1 \\ v^* \kappa\delta u(\beta, c) \end{pmatrix},$$

where $T(\beta, c) = G_u(u(\beta, c), \beta, c)$. Then $\mathscr{F}(\beta_0, c_0, v_2^*, 0, 0) = 0$ and

$$\mathscr{F}_{(v^*, \lambda, \sigma)}(\beta_0, c_0, v_2^*, 0, 0) = \begin{pmatrix} T_0 & -v_2^* K & -v_1^* K \\ Ku_2 & 0 & 0 \\ Ku_1 & 0 & 0 \end{pmatrix} \tag{2.14}$$

is an isomorphism. Thus, by choosing $\mathscr{B}_0$ and $\mathbf{R}_0$ smaller if necessary, we have by the implicit function theorem that there exists continuous functions $v^* \colon \mathscr{B}_0 \times \mathbf{R}_0 \to V^* - \{0\}$, $\lambda \colon \mathscr{B}_0 \times \mathbf{R}_0 \to \mathbf{R}$, and $\sigma \colon \mathscr{B}_0 \times \mathbf{R}_0 \to \mathbf{R}$ such that $v^*(\beta_0, c_0) = v_2^*$, $\lambda(\beta_0, c_0) = 0$, $\sigma(\beta_0, c_0) = 0$, and, for all $(\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0$,

$$\mathscr{F}(\beta, c, v^*(\beta, c), \lambda(\beta, c), \sigma(\beta, c)) = 0. \tag{2.15}$$

Suppose $(\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0$ and $\phi(\beta, c) = 0$. Then $G(u(\beta, c), \beta, c) = 0$. Hence

$$\kappa\delta u(\beta, c) = K\mathscr{D}_x u(\beta, c) \tag{2.16}$$

and $T(\beta, c)\, \mathscr{D}_x u(\beta, c) = 0$. So, by the last component of Eq. (2.15), $v^*(\beta, c)\, K\mathscr{D}_x u(\beta, c) = 0$. Thus, applying the first component of both sides of (2.15) to $\mathscr{D}_x u(\beta, c)$ we get

$$0 = \sigma(\beta, c)\, v_1^* K\mathscr{D}_x u(\beta, c) = \sigma(\beta, c)\, v_1^* \kappa\delta u(\beta, c).$$

Hence, since

$$\lim_{(\beta,c)\to(0,0)} v_1^* \kappa\delta u(\beta, c) = v_1^* \kappa\delta u_0 = v_1^* K u_1 = 1,$$

by taking $\mathscr{B}_0 \times \mathbf{R}_0$ smaller if necessary, we have, for $(\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0$ and $\phi(\beta, c) = 0$, that

$$\sigma(\beta, c) = 0 \tag{2.17}$$

and thus $(T(\beta, c) - \lambda(\beta, c)K)$: $U \to V$ is not onto V.

Note that the hypotheses of Theorem 0.1 hold. Let r and Q be as in Theorem 0.1. By choosing $\mathscr{B}_0 \times \mathbf{R}_0$ smaller if necessary, we have, for $(\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0$, that $|\lambda(\beta, c)| < r$, and by Theorem 0.1, for $|\lambda| < r$, that $R(Q(T(\beta, c))K)$ is 2-dimensional;

$$(T(\beta, c) - \lambda K)\colon U \to V \tag{2.18}$$

is an isomorphism if and only if

$$(Q(T(\beta, c))\,T(\beta, c) - \lambda I)\colon R(Q(T(\beta, c))K) \to R(Q(T(\beta, c))K) \tag{2.19}$$

is an isomorphism; and λ is a K-simple eigenvalue of

$$T(\beta, c)\colon U \to V \tag{2.20}$$

if and only if λ is an I-simple eigenvalue of

$$Q(T(\beta, c))\,T(\beta, c)\colon R(Q(T(\beta, c))K) \to R(Q(T(\beta, c))K). \tag{2.21}$$

Suppose $(\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0$ and $\phi(\beta, c) = 0$. Since (2.18) is not an isomorphism for $\lambda = 0$ or $\lambda = \lambda(\beta, c)$, the same is true of (2.19). Since $R(Q(T(\beta, c))K)$ is 2-dimensional, 0 and $\lambda(\beta, c)$ are eigenvalues of (2.21). To prove Theorem 2.2 part (i) it suffices to prove that 0 and $\lambda(\beta, c)$ are the only eigenvalues of (2.21). Since $R(Q(T(\beta, c))K)$ is 2-dimensional, the only way this couldn't be true is for $\lambda(\beta, c) = 0$ and for (2.21) to have a nonzero eigenvalue. But then zero would be an I-simple eigenvalue of (2.21) and hence a K-simple eigenvalue of (2.20). Thus we would have

$$V = R(T(\beta, c)) \oplus K\mathscr{D}_x u(\beta, c). \tag{2.22}$$

But by (2.15), (2.17), and (2.16) we have $v^*(\beta, c)\,T(\beta, c) = 0$ and $v^*(\beta, c)\,\mathscr{D}_x u(\beta, c) = 0$. Thus, by (2.22), we have $v^*(\beta, c) = 0$, which contradicts (2.15). So Theorem 2.2 part (i) is true. Theorem 2.2 part (iii) follows from the fact that the two eigenvalues of (2.21) are I-simple if and only if they are distinct.

Suppose $(\beta, c) \in \mathscr{B}_0 \times \mathbf{R}_0$ and $\phi(\beta, c) = 0$. Applying $v^*(\beta, c)$ to the first component,

$$T(\beta, c)\, u_c(\beta, c) - \phi_c(\beta, c)\, Ku_2 = \kappa\delta u(\beta, c),$$

of (2.13) and using (2.15) and (2.17) we get

$$\lambda(\beta, c)\, v^*(\beta, c)\, Ku_c(\beta, c) = \phi_c(\beta, c).$$

Hence, since

$$\lim_{(\beta, c) \to (\beta_0, c_0)} v^*(\beta, c)\, Ku_c(\beta, c) = v_2^* Ku_2 = 1,$$

we have, by choosing $\mathscr{B}_0 \times \mathbf{R}_0$ smaller if necessary, that Theorem 2.2 part (ii) holds.

Differentiating the first component of both sides of (2.12) with respect to β and evaluating at $(\beta, c) = (\beta_0, c_0)$ we get

$$T_0 u_\beta(\beta_0, c_0) - (Ku_2)\, \phi_\beta(\beta_0, c_0) = -G_\beta(u_0, \beta_0, c_0).$$

Thus, by (2.10), we have $R(G_\beta(u_0, \beta_0, c_0)) \subset R(T_0)$ if and only if $\phi_\beta(\beta_0, c_0) = 0$. Hence the last sentence of Theorem 2.2 follows from the implicit function theorem.

3. Proof of Theorems 1.1 and 1.2

We will now use Theorems 2.1 and 2.2 to prove Theorems 1.1 and 1.2. To do this we must first define $A, B, U, V, K, \kappa\delta$, and g such that Hypothesis (H) holds. Let $\mathscr{B}, n, p, F, E, D$, and $\mathscr{F}$ be as in Section 1. If $u: \mathbf{R} \to \mathbf{R}^n$ is a function then we define the components $u_1: \mathbf{R} \to \mathbf{R}^p$ and $u_2: \mathbf{R} \to \mathbf{R}^{n-p}$ of u by $u = \binom{u_1}{u_2}$. Let $A = B = \mathbf{R}^n$. Let U be the vector space of all continuously differentiable functions $u: \mathbf{R} \to \mathbf{R}^n$ such that u_1 is C^2 and $u(x)$, $u'(x)$, and $u_1''(x)$ all tend to zero as $|x| \to \infty$. Let V be the vector space of all continuous functions $v: \mathbf{R} \to \mathbf{R}^n$ such that $v(x)$ tends to zero as $|x| \to \infty$. With the norms

$$\|u\|_U = \sup_{x \in \mathbf{R}} \left\| \begin{pmatrix} u(x) \\ u_1'(x) \end{pmatrix} \right\|_{\mathbf{R}^{n+p}} + \sup_{x \in \mathbf{R}} \left\| \begin{pmatrix} u(x) \\ u_1'(x) \end{pmatrix}' \right\|_{\mathbf{R}^{n+p}}$$

and

$$\|u\|_V = \sup_{x \in \mathbf{R}} \|v(x)\|_{\mathbf{R}^n}$$

U and V are Banach spaces. Let $K: U \to V$ be the inclusion of U into V and define $\kappa\delta: U \to V$, $g: U \times \mathscr{B} \to V$, and $G: U \times \mathscr{B} \times \mathbf{R} \to V$ by $\kappa\delta u = u'$, $g(u, \beta)(x) = Eu''(x) + F(u(x), \beta)$, and $G(u, \beta, c) = g(u, \beta) - c\kappa\delta u$.

Suppose (u, β, c) is in the zero set of G. Let $y = (u_1, u_2, u_1')^T$. Then y is continuously differentiable and

$$y' = \mathscr{F}(y, \beta, c), \quad \lim_{|x| \to \infty} y(x) = 0. \tag{3.1}$$

Since F is twice continuously differentiable, so is $\mathscr{F}$. Thus, by (3.1), it follows that y, y', y'', and y''' exist and are continuous on $-\infty < x < \infty$ and tend to zero as $|x| \to \infty$. Hence by

$$\frac{y(x+\xi) - y(x)}{\xi} - y'(x) = \frac{1}{\xi} \int_x^{x+\xi} (x + \xi - \zeta)\, y''(\zeta)\, d\zeta$$

and the same formula with y replaced with y' we have that part (iii) of Hypothesis (H) holds. Clearly the other parts of Hypothesis (H) also hold.

Theorem 1.1 (resp. Theorem 1.2) will follow from Theorem 2.1 (resp. Theorem 2.2) once we show that the hypotheses of Theorem 1.1 (resp. Theorem 1.2) imply Hypothesis H_3 (resp. Hypothesis H_4). Assume the hypotheses of Theorem 1.1 (resp. Theorem 1.2). Clearly (u_0, β_0, c_0) is in the zero set of G, $\mathscr{D}_x u_0 \neq 0$, and, by Lemma 1.1, $N(T_0) = \operatorname{span}\{\mathscr{D}_x u_0\}$ where $T_0 = G_u(u_0, \beta_0, c_0)$.

The only remaining part of Hypothesis H_3 (resp. Hypothesis H_4) that is not obvious is that $R(T_0)$ has codimension one in V. We now prove this and in so doing complete the proof of Theorem 1.1 (resp. Theorem 1.2). Define $L: U \to V$ by

$$L\phi = E\phi'' - c\phi' + F_u(0, \beta_0)\phi.$$

Let Y be the vector space of all continuously differentiable functions $y: \mathbf{R} \to \mathbf{R}^{n+p}$ such that $y(x)$ and $y'(x)$ both tend to zero as $|x| \to \infty$. Let W be the vector space of all continuous functions $w: \mathbf{R} \to \mathbf{R}^{n+p}$ such that $w(x) \to 0$ as $|x| \to \infty$. With the norms

$$\|y\|_Y = \sup_{x \in \mathbf{R}} \|y(x)\|_{\mathbf{R}^{n+p}} + \sup_{x \in \mathbf{R}} \|y'(x)\|_{\mathbf{R}^{n+p}}$$

$$\|w\|_W = \sup_{x \in \mathbf{R}} \|w(x)\|_{\mathbf{R}^{n+p}}$$

Y and W are Banach spaces. Define $\mathscr{L}: Y \to W$ by $\mathscr{L}y = y' + My$, where $M = -\mathscr{F}_y(0, \beta_0, c_0)$. Then for $\phi \in U$ and $g \in V$ we have $L\phi = g$ if and only if

$$\mathscr{L}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_1' \end{pmatrix} = \begin{pmatrix} 0 \\ -(1/c)\, g_2 \\ D^{-1} g_1 \end{pmatrix}. \tag{3.2}$$

Since M has no eigenvalues on the imaginary axis, $\mathscr{L}$ is an isomorphism. Thus L is one to one. Let $g \in V$. Let y be the unique function in Y whose image under $\mathscr{L}$ is the right side of (3.2). Then $y = (\phi_1, \phi_2, \phi_1')^T$ for some $\phi \in U$. Thus $L\phi = g$. So L is onto V. Hence L is a Fredholm operator of index zero. Since, for $\phi \in U$,

$$(T_0 - L)\phi = (F_u(u_0(x), \beta_0) - F_u(0, \beta_0))\phi$$

it is easy to check, using the Arzela–Ascoli theorem, that T_0 differs from L by an L-compact operator. Thus (see [11, Chap. 4, Theorem 5.26]) T_0 is a Fredholm operator of index zero. Hence, since the null space of T_0 is one dimensional, we have the range of T_0 has codimension one in V.

Appendix

The main result of this appendix is Theorem 0.1 which is needed for the proof of Theorem 2.2. Theorem 0.1 follows easily from Lemmas 0.1 and 0.2. Since Lemma 0.2 can be proved by making only minor modifications in [11, Chaps. 1 and 2], we have omitted its proof. However, in Lemma 0.1, since zero is not a K-semisimple eigenvalue of T_0, Lemma 0.1 and its proof are new.

LEMMA 0.1. *Suppose U and V are real Banach spaces, $T_0, K\colon U \to V$ are bounded linear operators, $R(T_0)$ has codimension one in V, and for some $u_1, u_2 \in U$ we have $N(T_0) = \operatorname{span}\{u_1\}$, $T_0 u_2 = K u_1$, and $K u_2 \notin R(T_0)$. Then*

(i) *there exists $r > 0$ such that $(T_0 - \lambda K)\colon U \to V$ is an isomorphism for all complex numbers λ with $0 < |\lambda| \leqslant r$;*

(ii) *there exist unique bounded linear functionals $v_1^*, v_2^*\colon V \to \mathbf{R}$ such that $v_2^* T_0 = 0$, $v_1^* T_0 = v_2^* K$, and $v_j^* K u_k = \delta_{jk}$ for $j, k \in \{1, 2\}$;*

(iii) *$P\colon V \to U$ given by $Pv = (v_1^* v)\, u_1 + (v_2^* v)\, u_2$ is the unique bounded linear operator from V into U such that $PKP = P$, $T_0 PK = KPT_0$, the restriction*

$$(T_0 - \lambda K)\colon R(PK) \to R(KP)$$

of $T_0 - \lambda K$ is an isomorphism for all $\lambda \in \mathbb{C} - \{0\}$, and the restriction

$$T_0\colon N(PK) \to N(KP)$$

of T_0 is an isomorphism.

Proof. First we prove (ii). Since the codimension of $R(T_0)$ is finite we have $R(T_0)$ is a closed subspace of V; (see [11, Chap. 4, Sect. 5]). Also $Ku_2 \neq 0$ and

$$V = R(T_0) \oplus \operatorname{span}\{Ku_2\}. \tag{0.1}$$

Thus there exists a unique bounded linear functional $v_2^*: V \to \mathbf{R}$ such that $v_2^* T_0 = 0$ and $v_2^* K u_2 = 1$. Since $Ku_1 = T_0 u_2$ we have $v_2^* K u_1 = 0$.

Define $\hat{T}: U/N(T_0) \to R(T_0)$ by $\hat{T}[u] = T_0 u$. Then $\hat{T}$ is a well defined bounded isomorphism and hence, by the open mapping theorem, has a bounded inverse. Since $N(T_0) \subset N(v_2^* K)$ we have $\widehat{v_2^* K}: U/N(T_0) \to \mathbf{R}$ given by $\widehat{v_2^* K}[u] = v_2^* K u$ is a well defined bounded linear functional. Define $v_1^*: V \to \mathbf{R}$ by $v_1^* \mid R(T_0) = \widehat{v_2^* K} \hat{T}^{-1}$ and $v_1^* K u_2 = 0$ and extend linearly to V. Then v_1^* is a bounded linear functional. Clearly $v_1^* T_0 = v_2^* K$ and thus

$$v_1^* K u_1 = v_1^* T_0 u_2 = v_2^* K u_2 = 1.$$

By (0.1), v_1^* is unique.

To prove that P, as given in (iii), satisfies the properties given in (iii), it suffices to observe that each statement in the following set of statements follows directly from (ii) or a statement in the set that precedes it: $PKu_1 = u_1$ and $PKu_2 = u_2$; $PKP = P$; $T_0 PK = KPT_0$; PK and KP are projections; T_0 and K both map $R(PK)$ into $R(KP)$ and both map $N(PK)$ into $N(KP)$, we denote these restrictions by T_1 and K_1 and by T_2 and K_2, respectively; $R(PK) = \operatorname{span}\{u_1, u_2\}$, $R(KP) = \operatorname{span}\{Ku_1, Ku_2\}$, and $\{Ku_1, Ku_2\}$ is an independent set; $R(T_0) \supset N(KP)$; T_2 is an isomorphism; K_1 is an isomorphism;

$$K_1^{-1}(T_1 - \lambda K_1) = K_1^{-1} T_1 - \lambda I;$$

$K_1^{-1} T_1$ maps u_1 to zero and u_2 to u_1; zero is the only eigenvalue of $K_1^{-1} T_1$; $T_1 - \lambda K_1$ is an isomorphism for $\lambda \neq 0$.

To prove (i), note that since T_2 is an isomorphism it has, by the open mapping theorem, a bounded inverse and hence, for $|\lambda| < \|T_2^{-1} K_2\|^{-1}$,

$$T_2 - \lambda K_2 = T_2(I - \lambda T_2^{-1} K_2)$$

is an isomorphism. Thus, since $T_1 - \lambda K_1$ is an isomorphism for $\lambda \neq 0$, we have $T_0 - \lambda K$ is an isomorphism for $0 < |\lambda| < \|T_2^{-1} K_2\|^{-1}$.

Finally, we prove the uniqueness of P. Suppose $Q: V \to U$ were another bounded linear operator satisfying the conditions in (iii). Observe that each statement in the following set of statements follows from the assumptions on Q, or a statement in the set that precedes it: QK and KQ are projections; T_0 and K both map $R(QK)$ into $R(KQ)$ and both map $N(QK)$ into $N(KQ)$, we denote these restrictions by $\hat{T}_1$ and $\hat{K}_1$ and by $\hat{T}_2$ and $\hat{K}_2$, respectively; $\hat{T}_2$ is an isomorphism;

$$\hat{K}_1(R(QK)) = (KQK)(U) \supset (KQKQ)(V) = KQ(V) = R(KQ);$$

if $QKu \in N(\hat{K}_1)$ then $QKu = QKQKu = Q(0) = 0$; $\hat{K}_1$ is an isomorphism; $u_1 \in R(QK)$; $T_0 u_2 = Ku_1 \in R(KQ)$; $u_2 \in R(QK)$; $\text{span}\{u_1, u_2\} \subset R(QK)$;

We now show $\text{span}\{u_1, u_2\} = R(QK)$. By part (ii) of Lemma 0.1 applied to $\hat{T}_1, \hat{K}_1 : R(QK) \to R(KQ)$ there exist bounded linear functionals $\hat{v}_1^*, \hat{v}_2^* : R(KQ) \to \mathbf{R}$ such that $\hat{v}_2^* \hat{T}_1 = 0$, $\hat{v}_1^* \hat{T}_1 = \hat{v}_2^* \hat{K}_1$, and $\hat{v}_j^* \hat{K}_1 u_k = \delta_{jk}$ for $j = 1, 2$. Let $\hat{P}$: $R(KQ) \to R(QK)$ be given by

$$\hat{P}v = (\hat{v}_1^* v)\, u_1 + (\hat{v}_2^* v)\, u_2.$$

Then, as above, $\hat{P}\hat{K}_1\hat{P} = \hat{P}$, $\hat{T}_1\hat{P}\hat{K}_1 = \hat{K}_1\hat{P}\hat{T}_1$, and the restriction $\hat{T}_1 : N(\hat{P}\hat{K}_1) \to N(\hat{K}_1\hat{P})$ of $\hat{T}_1$ is an isomorphism. Thus, since, by assumption, $\hat{T}_1 - \lambda\hat{K}_1$ is an isomorphism for $\lambda \neq 0$ we have the restriction $(\hat{T}_1 - \lambda\hat{K}_1)$: $N(\hat{P}\hat{K}_1) \to N(\hat{K}_1\hat{P})$ of $\hat{T}_1 - \lambda\hat{K}_1$ is an isomorphism for all λ. Hence the bounded operator

$$\hat{K}_1^{-1}(\hat{T}_1 - \lambda\hat{K}_1) = (\hat{K}_1^{-1}\hat{T}_1 - \lambda I)$$

restricted to $N(\hat{P}\hat{K}_1)$ is an isomorphism for all λ. Therefore, since every bounded operator on a Banach space of nonzero dimension has a non-empty spectrum (see [11, p.176]), $N(\hat{P}\hat{K}_1) = \{0\}$. Hence the projection $\hat{P}\hat{K}_1$ is the identity on $R(QK)$. So

$$\text{span}\{u_1, u_2\} = R(\hat{P}\hat{K}_1) = R(QK) \tag{0.2}$$

and since $QKQ = Q$ we have

$$R(QK) = R(Q). \tag{0.3}$$

Now we show $Q = P$. By (0.2) and (0.3), for some bounded linear functionals y_1^* and y_2^* on V we have

$$Qv = (y_1^* v)u_1 + (y_2^* v)u_2. \tag{0.4}$$

Since QK is a projection and (0.2), we have for $k = 1, 2$ that

$$u_k = QKu_k = (y_1^* Ku_k)\, u_1 + (y_2^* Ku_k)\, u_2.$$

Hence $y_j^* Ku_k = \delta_{j,k}$ for $j,k \in \{1, 2\}$. Substituting (0.4) in $T_0 QK = KQT_0$ we get for all $u \in U$ that

$$(y_2^* Ku)\, Ku_1 = (y_1^* T_0 u)\, Ku_1 + (y_2^* T_0 u)\, Ku_2.$$

So $y_2^* K = y_1^* T_0$ and $y_2^* T_0 = 0$. So, by part (ii) of the lemma, $y_1^* = v_1^*$ and $y_2^* = v_2^*$. So $Q = P$.

LEMMA 0.2. *Suppose U and V are complex Banach spaces, K and T_0 are in the complex Banach space, $B(U, V)$, of all bounded linear operators from*

U into V, and, for some $r>0$, $(T_0-\lambda K)\colon U\to V$ is an isomorphism for $0<|\lambda|\leqslant r$. Then there exists $Q\colon B(U,V)\to B(V,U)$ which is continuous at T_0 such that for $\|T-T_0\|$ sufficiently small we have

(i) *$Q(T)KQ(T)=Q(T)$ (thus $KQ(T)\colon V\to V$ and $Q(T)K\colon U\to U$ are projections);*

(ii) *$KQ(T)T=TQ(T)K$ (thus both T and K map $R(Q(T)K)$ into $R(KQ(T))$ and map $N(Q(T)K)$ into $N(KQ(T))$);*

(iii) *$(T-\lambda K)\colon R(Q(T)K)\to R(KQ(T))$ is an isomorphism for $|\lambda|\geqslant r$;*

(iv) *$(T-\lambda K)\colon N(Q(T)K)\to N(KQ(T))$ is an isomorphism for $|\lambda|\leqslant r$.*

THEOREM 0.1. *Suppose the hypotheses in the first sentence of Lemma 0.1 hold. Let r and P be as in the conclusion of Lemma 0.1. Then there exists $Q\colon B(U,V)\to B(V,U)$, which is continuous at T_0, such that $Q(T_0)=P$ and for $\|T-T_0\|$ sufficiently small we have* (i)–(iv) *of Lemma 0.2 hold, $R(Q(T)K)$ is 2-dimensional, and, for $|\lambda|<r$,*

(i) *$(T-\lambda K)\colon U\to V$ is an isomorphism if and only if $(Q(T)T-\lambda I)\colon R(Q(T)K)\to R(Q(T)K)$ is an isomorphism;*

(ii) *λ is a K-simple eigenvalue of $T\colon U\to V$ if and only if λ is an I-simple eigenvalue of $Q(T)T\colon R(Q(T)K)\to R(Q(T)K)$.*

Proof. By Lemmas 0.1 and 0.2 there exists $Q\colon B(U,V)\to B(V,U)$ which is continuous at T_0 such that $Q(T_0)=P$ and for $\|T-T_0\|$ sufficiently small (i) – (iv) of Lemma 0.2 hold. By parts (ii) and (iii) of Lemma 0.1 we have $PKu_1=u_1$, $PKu_2=u_2$, and u_1 and u_2 are independent. Hence $R(PK)$ is 2-dimensional. Since $Q(T)K\colon U\to V$ is a projection and Q is continuous at T_0 we have by [11, Chap. I, Lemma 4.10] that $R(Q(T)K)$ is 2-dimensional for sufficiently small $\|T-T_0\|$. For sufficiently small $\|T-T_0\|$ we have by part (i) of Lemma 0.2 that

$$Q(T)K\colon R(Q(T)K)\to R(Q(T)K)$$

is the identity and

$$Q(T)\colon R(KQ(T))\to R(Q(T)K) \tag{0.5}$$

is an isomorphism. Hence parts (i) and (ii) of Theorem 0.1 follow from parts (iii) and (iv) of Lemma 0.2 by composing (0.5) with

$$(T-\lambda K)\colon R(Q(T)K)\to R(KQ(T)).$$

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